The conformal bootstrap in momentum space

Marc Gillioz

Hes·so // Valais Wallis
School of Engineering

University of Applied Sciences
& Arts of Western Switzerland

Sion, Switzerland

University of Haifa – April 30, 2025

Outline

What is special in conformal field theory?

Operator product expansion (OPE) for correlation functions:

$$\langle 0|\phi_1\phi_2\phi_3\phi_4|0\rangle = \sum_{|\psi\rangle} \langle 0|\phi_1\phi_2|\psi\rangle\langle\psi|\phi_3\phi_4|0\rangle$$

What is special in conformal field theory?

Operator product expansion (OPE) for correlation functions:

$$\langle 0|\phi_1\phi_2\phi_3\phi_4|0\rangle = \sum_{|\psi\rangle} \langle 0|\phi_1\phi_2|\psi\rangle\langle\psi|\phi_3\phi_4|0\rangle$$

- state/operator correspondence

What is special in conformal field theory?

Operator product expansion (OPE) for correlation functions:

$$\langle 0|\phi_1\phi_2\phi_3\phi_4|0\rangle = \sum_{|\psi\rangle} \langle 0|\phi_1\phi_2|\psi\rangle\langle\psi|\phi_3\phi_4|0\rangle$$

- state/operator correspondence
- distinguish primaries ($\mathcal{O}$) from descendants ($\partial\mathcal{O}$)

What is special in conformal field theory?

Operator product expansion (OPE) for correlation functions:

$$\langle 0|\phi_1\phi_2\phi_3\phi_4|0\rangle = \sum_{|\psi\rangle} \langle 0|\phi_1\phi_2|\psi\rangle\langle\psi|\phi_3\phi_4|0\rangle$$

- state/operator correspondence
- distinguish primaries ($\mathcal{O}$) from descendants ($\partial\mathcal{O}$)
- operators characterized by scaling dimension Δ and spin

What is special in conformal field theory?

Operator product expansion (OPE) for correlation functions:

$$\langle 0|\phi_1\phi_2\phi_3\phi_4|0\rangle = \sum_{|\psi\rangle} \langle 0|\phi_1\phi_2|\psi\rangle\langle\psi|\phi_3\phi_4|0\rangle$$

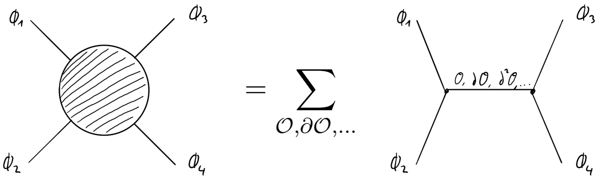
- state/operator correspondence
- distinguish primaries ($\mathcal{O}$) from descendants ($\partial\mathcal{O}$)
- operators characterized by scaling dimension Δ and spin
- 3-point functions fixed in terms of Δ_i and C_{ijk}

What is special in conformal field theory?

Operator product expansion (OPE) for correlation functions:

$$\langle 0|\phi_1\phi_2\phi_3\phi_4|0\rangle = \sum_{|\psi\rangle} \langle 0|\phi_1\phi_2|\psi\rangle\langle\psi|\phi_3\phi_4|0\rangle$$

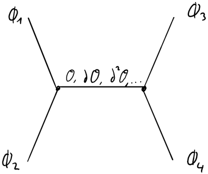
- state/operator correspondence
- distinguish primaries ($\mathcal{O}$) from descendants ($\partial\mathcal{O}$)
- operators characterized by scaling dimension Δ and spin
- 3-point functions fixed in terms of Δ_i and C_{ijk}



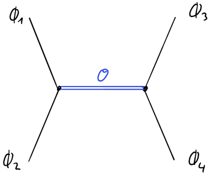
Conformal blocks in 1d

≡ resummed contribution of descendants

$$\sum_{\mathcal{O},\partial\mathcal{O},\ldots}$$



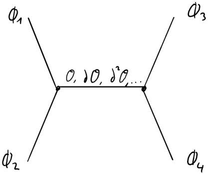
$$= \sum_{\mathcal{O}}$$



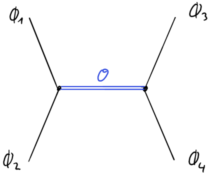
Conformal blocks in 1d

≡ resummed contribution of descendants

$$\sum_{\mathcal{O},\partial\mathcal{O},\ldots}$$



$$= \sum_{\mathcal{O}}$$

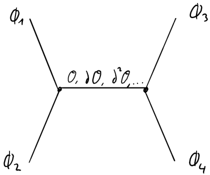


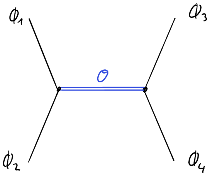
$$= \sum_{\mathcal{O}} C^2_{\phi\phi\mathcal{O}}\, z^{\Delta-2\Delta_\phi}\, {}_2F_1(\Delta,\Delta;2\Delta;z)$$

Cross-ratio: $z = \dfrac{(x_1-x_2)(x_3-x_4)}{(x_1-x_4)(x_3-x_2)} \in (0,1)$

Conformal blocks in 1d

$\equiv$ resummed contribution of descendants

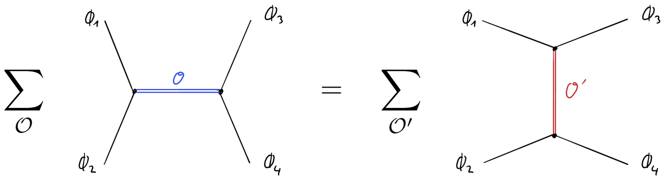
$\sum_{\mathcal{O},\partial\mathcal{O},\ldots}$ 

$= \sum_{\mathcal{O}}$ 

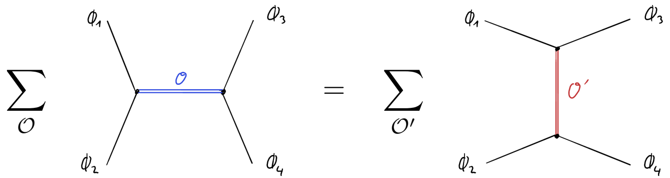
$$= \sum_{\mathcal{O}} C^2_{\phi\phi\mathcal{O}}\, z^{\Delta-2\Delta_\phi}\, {}_2F_1(\Delta,\Delta;2\Delta;z)$$

Cross-ratio: $z = \dfrac{(x_1-x_2)(x_3-x_4)}{(x_1-x_4)(x_3-x_2)}$ $\in \mathbb{C}\setminus(-\infty,0)\cup(1,\infty)$

Crossing symmetry



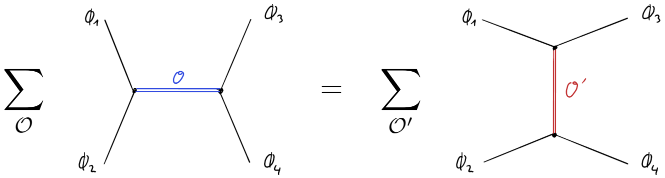
Crossing symmetry



The 1d bootstrap equation

$$\sum_{\mathcal{O}} C^2_{\phi\phi\mathcal{O}} z^{\Delta-2\Delta_\phi} {}_2F_1(\Delta,\Delta;2\Delta;z)$$
$$= \sum_{\mathcal{O}} C^2_{\phi\phi\mathcal{O}} (1-z)^{\Delta-2\Delta_\phi} {}_2F_1(\Delta,\Delta;2\Delta;1-z)$$

Crossing symmetry



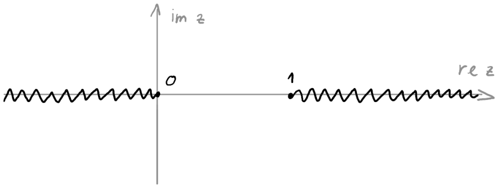
The 1d bootstrap equation

$$\sum_{\mathcal{O}} C^2_{\phi\phi\mathcal{O}} \Big[z^{\Delta-2\Delta_\phi} {}_2F_1(\Delta,\Delta;2\Delta;z) - (1-z)^{\Delta-2\Delta_\phi} {}_2F_1(\Delta,\Delta;2\Delta;1-z) \Big] = 0$$

Applying functionals

The 1d bootstrap equation

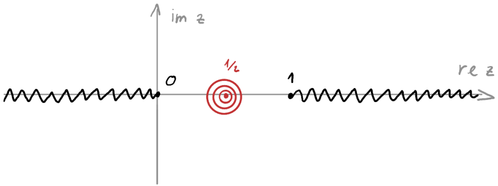
$$\sum_{\mathcal{O}} C^2_{\phi\phi\mathcal{O}} \Big[z^{\Delta-2\Delta_\phi} {}_2F_1(\Delta,\Delta;2\Delta;z) - (1-z)^{\Delta-2\Delta_\phi} {}_2F_1(\Delta,\Delta;2\Delta;1-z) \Big] = 0$$



Applying functionals

The 1d bootstrap equation

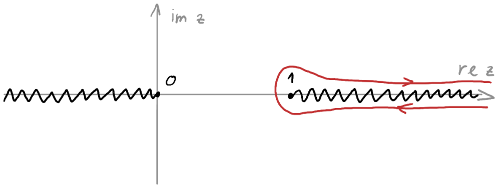
$$\sum_{\mathcal{O}} C^2_{\phi\phi\mathcal{O}} \Big[z^{\Delta-2\Delta_\phi} {}_2F_1(\Delta,\Delta;2\Delta;z) - (1-z)^{\Delta-2\Delta_\phi} {}_2F_1(\Delta,\Delta;2\Delta;1-z) \Big] = 0$$



Applying functionals

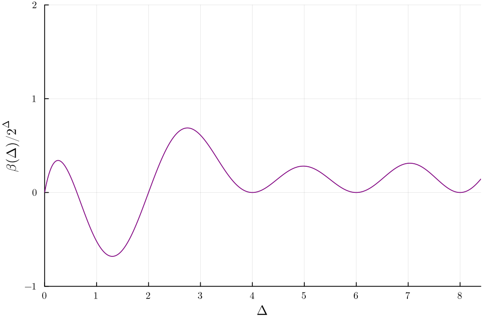
The 1d bootstrap equation

$$\sum_{\mathcal{O}} C^2_{\phi\phi\mathcal{O}} \Big[z^{\Delta-2\Delta_\phi} {}_2F_1(\Delta,\Delta;2\Delta;z) - (1-z)^{\Delta-2\Delta_\phi} {}_2F_1(\Delta,\Delta;2\Delta;1-z) \Big] = 0$$



$$\beta(\Delta) = \int_1^\infty dz\, f(z)\, \mathrm{im}[\ldots]$$

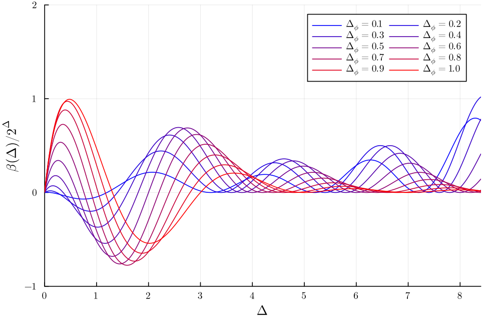
Optimal functionals



Example: $\Delta_\phi = \frac{1}{2}$: $\quad \beta(\Delta) \geq 0 \qquad \forall\ \Delta \geq 2$ Mazáč, 1611.10060

$$f(z) = 1 + \frac{1}{z(z-1)} - \left[\frac{z(2z^2-5z+5)}{2(z-1)^2}\log|z| + (z \leftrightarrow 1-z)\right]$$

Optimal functionals



Generalization: $\beta(0) = 0, \quad \beta(\Delta) \geq 0 \qquad \forall\ \Delta \geq 2\Delta_\phi + 1$

$\Longrightarrow$ bound on scaling dimension of lowest-lying operator

$$\Delta \leq 2\Delta_\phi + 1$$

The generalized free fermion theory

Zeros of optimal functionals correspond to actual operators:

- Identity (vacuum state) with $\Delta = 0$
- Wick contractions of anticommuting field $\mathcal{O}_n \sim \phi \Box^n \overleftrightarrow{\partial} \phi$ with $\Delta = 2\Delta_\phi + 2n + 1$ $(n \in \mathbb{N})$ and

$$C^2_{\phi\phi\mathcal{O}_n} = 2\frac{(2\Delta_\phi)^2_{2n+1}}{(2n+1)!(4\Delta_\phi + 2n)_{2n+1}}$$

Correlation function:

$$\langle 0|\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)|0\rangle \propto -1 + z^{-2\Delta_\phi} + (1-z)^{-2\Delta_\phi}$$

The generalized free fermion theory

Zeros of optimal functionals correspond to actual operators:

- Identity (vacuum state) with $\Delta = 0$
- Wick contractions of anticommuting field $\mathcal{O}_n \sim \phi \Box^n \overleftrightarrow{\partial} \phi$ with $\Delta = 2\Delta_\phi + 2n + 1$ $(n \in \mathbb{N})$ and

$$C^2_{\phi\phi\mathcal{O}_n} = 2 \frac{(2\Delta_\phi)^2_{2n+1}}{(2n+1)!(4\Delta_\phi + 2n + 1 - 1)_{2n+1}}$$

Correlation function:

$$\langle 0|\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)|0\rangle \propto \pm 1 + z^{-2\Delta_\phi} + (1-z)^{-2\Delta_\phi}$$

Works also with generalized free boson: $\Delta = 2\Delta_\phi + 2n$

The new crossing equation

Standard bootstrap equation $z \in \mathbb{C}$

$$\sum_{\mathcal{O}} C^2_{\phi\phi\mathcal{O}} \Big[z^{\Delta - 2\Delta_\phi} {}_2F_1(\Delta, \Delta; 2\Delta; z) - (1-z)^{\Delta - 2\Delta_\phi} {}_2F_1(\Delta, \Delta; 2\Delta; 1-z) \Big] = 0$$

New momentum-space bootstrap equation $w \in (0,1)$

$$\sum_{\mathcal{O}} C^2_{\phi\phi\mathcal{O}} \Big[\frac{\Gamma(2\Delta)}{\Gamma(\Delta)^2 \Gamma(2\Delta_\phi)} w^{2\Delta_\phi - 1} {}_2F_1(1-\Delta, \Delta; 2\Delta_\phi; w) - \frac{w^{\Delta-1}}{\Gamma(2\Delta_\phi - \Delta)\Gamma(\Delta)} {}_2F_1(\Delta - 2\Delta_\phi + 1, \Delta; 2\Delta; w) \Big] = 0$$

Comparison

Standard bootstrap equation

$$\sum_{\mathcal{O}} C^2_{\phi\phi\mathcal{O}} \Big[z^{\Delta-2\Delta_\phi} {}_2F_1(\Delta,\Delta;2\Delta;z) - (1-z)^{\Delta-2\Delta_\phi} {}_2F_1(\Delta,\Delta;2\Delta;1-z) \Big] = 0$$

New bootstrap equation

$$\sum_{\mathcal{O}} C^2_{\phi\phi\mathcal{O}} \Big[\frac{\Gamma(2\Delta)}{\Gamma(\Delta)^2\Gamma(2\Delta_\phi)} w^{2\Delta_\phi-1} {}_2F_1(1-\Delta,\Delta;2\Delta_\phi;w) - \frac{w^{\Delta-1}}{\Gamma(2\Delta_\phi-\Delta)\Gamma(\Delta)} {}_2F_1(\Delta-2\Delta_\phi+1,\Delta;2\Delta;w) \Big] = 0$$

- Symmetric $z \leftrightarrow 1-z$
- Asymmetric

Comparison

Standard bootstrap equation

$$\sum_{\mathcal{O}} C^2_{\phi\phi\mathcal{O}} \Big[z^{\Delta-2\Delta_\phi}{}_2F_1(\Delta,\Delta;2\Delta;z) - (1-z)^{\Delta-2\Delta_\phi}{}_2F_1(\Delta,\Delta;2\Delta;1-z) \Big] = 0$$

New bootstrap equation

$$\sum_{\mathcal{O}} C^2_{\phi\phi\mathcal{O}} \Big[\frac{\Gamma(2\Delta)}{\Gamma(\Delta)^2\Gamma(2\Delta_\phi)} w^{2\Delta_\phi-1}{}_2F_1(1-\Delta,\Delta;2\Delta_\phi;w) - \frac{w^{\Delta-1}}{\Gamma(2\Delta_\phi-\Delta)\Gamma(\Delta)}{}_2F_1(\Delta-2\Delta_\phi+1,\Delta;2\Delta;w) \Big] = 0$$

- Symmetric $z \leftrightarrow 1-z$
- Absolute convergence for $z \in \mathbb{C} \setminus (-\infty,0) \cup (1,\infty)$

- Asymmetric
- No absolute convergence in s-channel

Comparison

Standard bootstrap equation

$$\sum_{\mathcal{O}} C^2_{\phi\phi\mathcal{O}} \Big[z^{\Delta-2\Delta_\phi} {}_2F_1(\Delta,\Delta;2\Delta;z) - (1-z)^{\Delta-2\Delta_\phi} {}_2F_1(\Delta,\Delta;2\Delta;1-z) \Big] = 0$$

- Symmetric $z \leftrightarrow 1-z$
- Absolute convergence for $z \in \mathbb{C} \setminus (-\infty,0) \cup (1,\infty)$
- Blocks are unphysical

New bootstrap equation

$$\sum_{\mathcal{O}} C^2_{\phi\phi\mathcal{O}} \Big[\frac{\Gamma(2\Delta)}{\Gamma(\Delta)^2\Gamma(2\Delta_\phi)} w^{2\Delta_\phi-1} {}_2F_1(1-\Delta,\Delta;2\Delta_\phi;w) - \frac{w^{\Delta-1}}{\Gamma(2\Delta_\phi-\Delta)\Gamma(\Delta)} {}_2F_1(\Delta-2\Delta_\phi+1,\Delta;2\Delta;w) \Big] = 0$$

- Asymmetric
- No absolute convergence in s-channel
- Blocks are 3-pt functions in momentum space

$\Downarrow$

peculiar properties

Features of the new equation

$$\sum_{\mathcal{O}} C^2_{\phi\phi\mathcal{O}} \left[\frac{\Gamma(2\Delta)}{\Gamma(\Delta)^2\Gamma(2\Delta_\phi)} w^{2\Delta_\phi - 1} {}_2F_1(1-\Delta, \Delta; 2\Delta_\phi; w) - \frac{w^{\Delta-1}}{\Gamma(2\Delta_\phi - \Delta)\Gamma(\Delta)} {}_2F_1(\Delta - 2\Delta_\phi + 1, \Delta; 2\Delta; w) \right] = 0$$

Identity operator: $\Delta = 0$

- s channel is zero
- t channel $\propto \delta(w)$

Features of the new equation

$$\sum_{\mathcal{O}} C^2_{\phi\phi\mathcal{O}} \left[\frac{\Gamma(2\Delta)}{\Gamma(\Delta)^2 \Gamma(2\Delta_\phi)} w^{2\Delta_\phi - 1} {}_2F_1(1-\Delta, \Delta; 2\Delta_\phi; w) - \frac{w^{\Delta-1}}{\Gamma(2\Delta_\phi - \Delta)\Gamma(\Delta)} {}_2F_1(\Delta - 2\Delta_\phi + 1, \Delta; 2\Delta; w) \right] = 0$$

Identity operator: $\Delta = 0$

- s channel is zero
- t channel $\propto \delta(w)$

Double-twist operators: $\Delta = 2\Delta_\phi + n$ with $n \in \mathbb{N}$

- t channel is zero
- s channel is Gegenbauer polynomial $C_n^{(2\Delta_\phi - 1/2)}(1-2w)$

Features of the new equation

$$\sum_{\mathcal{O}} C^2_{\phi\phi\mathcal{O}} \Big[\frac{\Gamma(2\Delta)}{\Gamma(\Delta)^2\Gamma(2\Delta_\phi)} w^{2\Delta_\phi - 1} {}_2F_1(1-\Delta,\Delta;2\Delta_\phi;w) - \frac{w^{\Delta-1}}{\Gamma(2\Delta_\phi-\Delta)\Gamma(\Delta)} {}_2F_1(\Delta-2\Delta_\phi+1,\Delta;2\Delta;w)\Big] = 0$$

Identity operator: $\Delta = 0$

- s channel is zero
- t channel $\propto \delta(w)$

Double-twist operators: $\Delta = 2\Delta_\phi + n$ with $n \in \mathbb{N}$

- t channel is zero
- s channel is Gegenbauer polynomial $C_n^{(2\Delta_\phi - 1/2)}(1-2w)$

Other special case: $\Delta \in \mathbb{N}$

- s channel is Jacobi polynomial $P_n^{(2\Delta_\phi - 1, 1 - 2\Delta_\phi)}(1-2w)$

Connection between the two equations

Projection onto a basis of Jacobi polynomials

$$\int_0^1 dw (1-w)^{1-2\Delta_\phi} P_n(1-2w)[\ldots] \qquad \forall\, n \in \mathbb{N}$$

New bootstrap equation projected (1)

$$\sum_{\mathcal{O}} C^2_{\phi\phi\mathcal{O}} \left[\frac{\Gamma(2\Delta)}{\Gamma(\Delta)^2} \frac{\sin(\pi\Delta)}{\pi(\Delta+n)(1-\Delta+n)} - \frac{\sin[\pi(2\Delta_\phi-\Delta)](2\Delta_\phi-\Delta)_n}{\pi(\Delta-2\Delta_\phi+1)_{n+1}} \right.$$
$$\left. \times\, {}_4F_3\left(\begin{matrix} \Delta, \Delta, \Delta-2\Delta_\phi+1, \Delta-2\Delta_\phi+1 \\ 2\Delta, \Delta-2\Delta_\phi+n+2, \Delta-2\Delta_\phi-n+1 \end{matrix} ; 1 \right) \right] = 0$$

Connection between the two equations

Projection onto a basis of Jacobi polynomials

$$\int_0^1 dw(1-w)^{1-2\Delta_\phi} P_n(1-2w)[\ldots] \qquad \forall\, n \in \mathbb{N}$$

New bootstrap equation projected (1)

$$\sum_{\mathcal{O}} C^2_{\phi\phi\mathcal{O}} \left[\frac{\Gamma(2\Delta)}{\Gamma(\Delta)^2} \frac{\sin(\pi\Delta)}{\pi(\Delta+n)(1-\Delta+n)} - \frac{\sin[\pi(2\Delta_\phi-\Delta)](2\Delta_\phi-\Delta)_n}{\pi(\Delta-2\Delta_\phi+1)_{n+1}} \times {}_4F_3\left(\begin{matrix}\Delta,\Delta,\Delta-2\Delta_\phi+1,\Delta-2\Delta_\phi+1 \\ 2\Delta,\Delta-2\Delta_\phi+n+2,\Delta-2\Delta_\phi-n+1\end{matrix};1\right)\right] = 0$$

A known basis of functionals!!! Mazáč, 1611.10060

$\Rightarrow$ new equation is integral transform of original along branch cut

A nice basis of functionals

New bootstrap equation projected (2) $\forall\, n \in \mathbb{N}$

$$\sum_{\mathcal{O}} C^2_{\phi\phi\mathcal{O}} \Bigg[\frac{\Gamma(2\Delta)}{\Gamma(\Delta)^2\Gamma(2\Delta_\phi-\Delta)\Gamma(\Delta+2\Delta_\phi-1)} \times \frac{1}{(2\Delta_\phi-\Delta+n)(2\Delta_\phi+\Delta-1+n)} - \frac{\Gamma(\Delta-2\Delta_\phi+1)}{\Gamma(2\Delta_\phi-\Delta)\Gamma(\Delta-2\Delta_\phi+1-n)\Gamma(\Delta+2\Delta_\phi+n)} \times {}_4F_3\left(\begin{matrix} \Delta, \Delta, \Delta-2\Delta_\phi+1, \Delta+2\Delta_\phi-1 \\ 2\Delta, \Delta-2\Delta_\phi+1-n, \Delta+2\Delta_\phi+n \end{matrix} ;1 \right) \Bigg] = 0$$

- Zero at all double twist dimension but $\Delta = 2\Delta_\phi + n$
- Known as contour integral in cross-ratio space

  Mazáč, Rastelli, Zhou 1910.12855

- Convenient for constructing free boson/fermion functionals

Generalized free field solution

Plug in scaling dimensions:

- identity $\Delta = 0$ (only t channel)
- double-twist operators $\Delta = 2\Delta_\phi + k,\ \ k \in \mathbb{N}$ (only s channel)

$$\frac{(-1)^n}{\Gamma(2\Delta_\phi)^2}\left[C^2_{\phi\phi\mathcal{O}_n}\frac{n!(4\Delta_\phi+n-1)_n}{(2\Delta_\phi)_n^2}-1\right]=0$$

Recover free fermion/boson OPE coefficients for odd/even n

$$C^2_{\phi\phi\mathcal{O}_n}=\frac{(2\Delta_\phi)_n^2}{n!(4\Delta_\phi+n-1)_n}$$

How did we derive the new equation?

Simple Fourier transform?

$$\widetilde{\phi}(p) = \int d^d x \, e^{ip\cdot x} \phi(x)$$

In Euclidean space:

- correlators ill-defined at coincident points
- Fourier transform does not commute with OPE

How did we derive the new equation?

Simple Fourier transform?

$$\widetilde{\phi}(p) = \int d^d x \, e^{ip\cdot x} \phi(x)$$

In Euclidean space:

- correlators ill-defined at coincident points
- Fourier transform does not commute with OPE

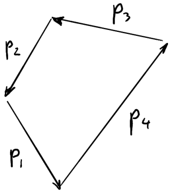
In Minkowski space:

- time-ordered correlators do not admit an OPE
- OPE for Wightman functions, but no crossing symmetry

$\Rightarrow$ must be replaced by something else

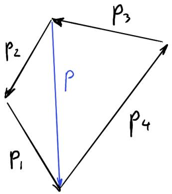
Momentum-space OPE

$$\langle 0|\widetilde{\phi}(p_1)\widetilde{\phi}(p_2)\widetilde{\phi}(p_3)\widetilde{\phi}(p_4)|0\rangle$$
$$= \delta^d(p_1+p_2+p_3+p_4)\langle\langle\widetilde{\phi}(p_1)\widetilde{\phi}(p_2)\widetilde{\phi}(p_3)\widetilde{\phi}(p_4)\rangle\rangle$$



Momentum-space OPE

$$\langle 0|\widetilde{\phi}(p_1)\widetilde{\phi}(p_2)\widetilde{\phi}(p_3)\widetilde{\phi}(p_4)|0\rangle$$
$$= \delta^d(p_1+p_2+p_3+p_4)\langle\langle\widetilde{\phi}(p_1)\widetilde{\phi}(p_2)\widetilde{\phi}(p_3)\widetilde{\phi}(p_4)\rangle\rangle$$

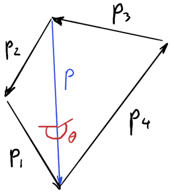


$$\langle\langle\widetilde{\phi}(p_1)\widetilde{\phi}(p_2)\widetilde{\phi}(p_3)\widetilde{\phi}(p_4)\rangle\rangle$$
$$\sim \sum_{\mathcal{O}} \langle\langle\widetilde{\phi}(p_1)\widetilde{\phi}(p_2)\mathcal{O}(-p)\rangle\rangle\langle\langle\mathcal{O}(p)\widetilde{\phi}(p_3)\widetilde{\phi}(p_4)\rangle\rangle$$

with $p = p_1 + p_2 = -p_3 - p_4$

Momentum-space OPE

$$\langle 0|\widetilde{\phi}(p_1)\widetilde{\phi}(p_2)\widetilde{\phi}(p_3)\widetilde{\phi}(p_4)|0\rangle$$
$$= \delta^d(p_1+p_2+p_3+p_4)\langle\!\langle\widetilde{\phi}(p_1)\widetilde{\phi}(p_2)\widetilde{\phi}(p_3)\widetilde{\phi}(p_4)\rangle\!\rangle$$



$$\langle\!\langle\widetilde{\phi}(p_1)\widetilde{\phi}(p_2)\widetilde{\phi}(p_3)\widetilde{\phi}(p_4)\rangle\!\rangle$$
$$\sim \sum_{\mathcal{O},m} \langle\!\langle\widetilde{\phi}(p_1)\widetilde{\phi}(p_2)\mathcal{O}^m(-p)\rangle\!\rangle\langle\!\langle\mathcal{O}^m(p)\widetilde{\phi}(p_3)\widetilde{\phi}(p_4)\rangle\!\rangle C_m^{(d-3)/2}(\cos\theta)$$

with $p = p_1 + p_2 = -p_3 - p_4$

and spin indices in $d > 2$

Momentum-space conformal blocks

Conformal blocks are products of 3-point functions:

- Generalized hypergeometric functions of ratios of momenta $\frac{p_i^2}{p_j^2}$

Momentum-space conformal blocks

Conformal blocks are products of 3-point functions:

- Generalized hypergeometric functions of ratios of momenta $\frac{p_i^2}{p_j^2}$
- Analytic except at light-cone crossings

$$\langle\!\langle \widetilde{\phi}_1(p_1)\widetilde{\phi}_2(p_2)\widetilde{\phi}_3(p_3)\rangle\!\rangle \propto (p_i^2)^{\Delta_i - d/2} \qquad \text{as} \quad p_i^2 \to 0_\pm$$

Momentum-space conformal blocks

Conformal blocks are products of 3-point functions:

- Generalized hypergeometric functions of ratios of momenta $\frac{p_i^2}{p_j^2}$
- Analytic except at light-cone crossings

$$\langle\!\langle \widetilde{\phi}_1(p_1)\widetilde{\phi}_2(p_2)\widetilde{\phi}_3(p_3)\rangle\!\rangle \propto (p_i^2)^{\Delta_i - d/2} \qquad \text{as} \quad p_i^2 \to 0_\pm$$

- Tempered distributions: singularities are integrable $(\Delta_i > \frac{d}{2} - 1)$

Momentum-space conformal blocks

Conformal blocks are products of 3-point functions:

- Generalized hypergeometric functions of ratios of momenta $\frac{p_i^2}{p_j^2}$
- Analytic except at light-cone crossings

$$\langle\!\langle \widetilde{\phi}_1(p_1)\widetilde{\phi}_2(p_2)\widetilde{\phi}_3(p_3)\rangle\!\rangle \propto (p_i^2)^{\Delta_i - d/2} \qquad \text{as} \quad p_i^2 \to 0_\pm$$

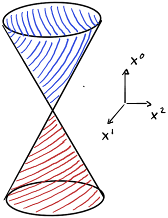
- Tempered distributions: singularities are integrable $(\Delta_i > \frac{d}{2} - 1)$
- Residue is ordinary ${}_2F_1$ hypergeometric function (two different forms depending on the limit)

Gillioz 2012.09825

The consequences of causality in momentum space

Causality: the commutator has support in the light cone

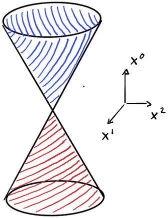
$$[\phi(0), \phi(x)] = 0 \qquad \forall \ |\vec{x}| \geq |x^0|$$



The consequences of causality in momentum space

Causality: the commutator has support in the light cone

$$[\phi(0), \phi(x)] = 0 \qquad \forall\ |\vec{x}| \geq |x^0|$$

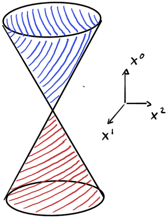


$\Rightarrow$ special analyticity properties in q for $\left[\widetilde{\phi}(p-q), \widetilde{\phi}(p+q)\right]$

The consequences of causality in momentum space

Causality: the commutator has support in the light cone

$$[\phi(0), \phi(x)] = 0 \qquad \forall \ |\vec{x}| \geq |x^0|$$



$\Rightarrow$ special analyticity properties in q for $\left[\widetilde{\phi}(p-q), \widetilde{\phi}(p+q)\right]$

$\Rightarrow$ Jost-Lehmann-Dyson representation for 4-pt function (1957)

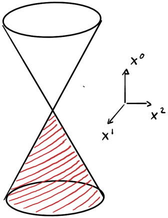
$$\langle 0|\widetilde{\phi}(-k-p)\left[\widetilde{\phi}(p-q), \widetilde{\phi}(p+q)\right]\widetilde{\phi}(k-p)|0\rangle$$

also valid without mass gap

The consequences of causality in momentum space

Causality: the retarded commutator has support in the past light cone

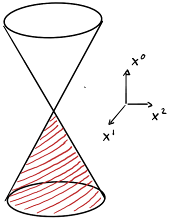
$$\theta(-x^0)\left[\phi(0), \phi(x)\right] = 0 \qquad \forall \ |\vec{x}| \geq -x^0$$



$\Rightarrow$ special analyticity properties in q for $\left[\widetilde{\phi}(p-q), \widetilde{\phi}(p+q)\right]$

The consequences of causality in momentum space

Causality: the retarded commutator has support in the past light cone



$$\theta(-x^0)\left[\phi(0),\phi(x)\right] = 0 \qquad \forall\ |\vec{x}| \geq -x^0$$

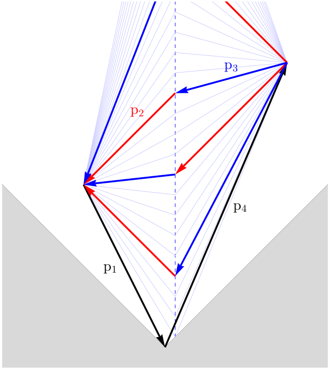
$\Rightarrow$ special analyticity properties in q for $\left[\widetilde{\phi}(p-q),\widetilde{\phi}(p+q)\right]$

$$\Rightarrow \quad \int dq^0 \frac{1}{q^0 - i} \langle 0|\widetilde{\phi}(-k-p)\left[\widetilde{\phi}(p-q),\widetilde{\phi}(p+q)\right]\widetilde{\phi}(k-p)|0\rangle$$

is **analytic** in $\vec{q}$ in the domain $|\mathrm{im}\,\vec{q}| < 1$

Discontinuities of the conformal blocks

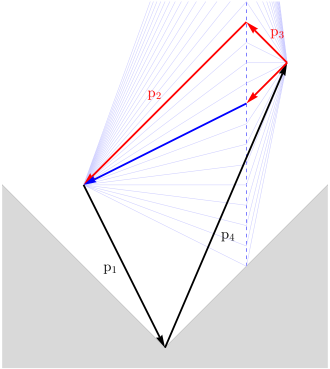
Examine analyticity in s and t channel & for each conformal block with p_1, p_4, $\vec{q}$ fixed and varying q^0, where $q = \frac{1}{2}(p_3 - p_2)$



The integral is analytic at generic $\vec{q}$

Discontinuities of the conformal blocks

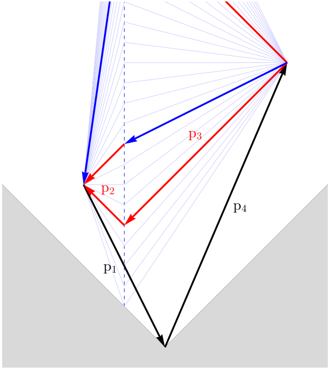
Examine analyticity in s and t channel & for each conformal block with p_1, p_4, $\vec{q}$ fixed and varying q^0, where $q = \frac{1}{2}(p_3 - p_2)$



$\exists\ \vec{q}_*$ such that the integral is discontinuous at $\vec{q} = \vec{q}_*$

Discontinuities of the conformal blocks

Examine analyticity in s and t channel & for each conformal block with p_1, p_4, $\vec{q}$ fixed and varying q^0, where $q = \frac{1}{2}(p_3 - p_2)$



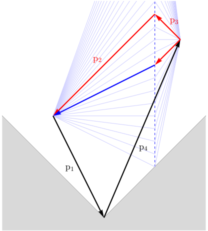
$\exists\ \vec{q}_*$ such that the integral is discontinuous at $\vec{q} = \vec{q}_*$ and $\vec{q} = -\vec{q}_*$

Matching discontinuities in 2d

Kinematics of discontinuity fixed by p_1 and p_4:

$$w \equiv -\frac{p_1^+}{p_4^+} = \frac{p_1^2}{s} \qquad \in (0,1)$$

$$\bar{w} \equiv -\frac{p_4^-}{p_1^-} = \frac{p_4^2}{s} \qquad \in (0,1)$$



$$\operatorname*{disc}_{\vec{q}=\vec{q}_*} \int \frac{dq^0}{q^0 - i} \langle\langle \widetilde{\phi}(p_1)\widetilde{\phi}(p-q)\widetilde{\phi}(p+q)\widetilde{\phi}(p_4)\rangle\rangle \propto \sum_{\mathcal{O}} C^2_{\phi\phi\mathcal{O}} S_h(w) S_{\bar{h}}(\bar{w})$$

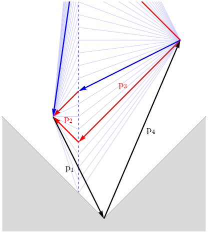
$$S_\Delta(w) = \frac{\Gamma(2\Delta)}{\Gamma(\Delta)^2\Gamma(2\Delta_\phi)} w^{2\Delta_\phi - 1} {}_2F_1(1-\Delta, \Delta; 2\Delta_\phi; w)$$

Matching discontinuities in 2d

Kinematics of discontinuity fixed by p_1 and p_4:

$$w \equiv -\frac{p_1^+}{p_4^+} = \frac{p_1^2}{s} = \frac{t}{p_4^2} \qquad \in (0,1)$$

$$\bar{w} \equiv -\frac{p_4^-}{p_1^-} = \frac{p_4^2}{s} = \frac{t}{p_1^2} \qquad \in (0,1)$$



$$\underset{\vec{q}=\vec{q}_*}{\text{disc}} \int \frac{dq^0}{q^0 - i} \langle\langle \widetilde{\phi}(p_1)\widetilde{\phi}(p-q)\widetilde{\phi}(p+q)\widetilde{\phi}(p_4)\rangle\rangle \propto \sum_{\mathcal{O}} C^2_{\phi\phi\mathcal{O}} S_h(w) S_{\bar{h}}(\bar{w})$$

$$\underset{\vec{q}=\vec{q}_*}{\text{disc}} \int \frac{dq^0}{q^0 - i} \langle\langle \widetilde{\phi}(p_1)\widetilde{\phi}(p+q)\widetilde{\phi}(p-q)\widetilde{\phi}(p_4)\rangle\rangle \propto \sum_{\mathcal{O}} C^2_{\phi\phi\mathcal{O}} T_h(w) T_{\bar{h}}(\bar{w})$$

$$S_\Delta(w) = \frac{\Gamma(2\Delta)}{\Gamma(\Delta)^2\Gamma(2\Delta_\phi)} w^{2\Delta_\phi - 1} {}_2F_1(1-\Delta, \Delta; 2\Delta_\phi; w)$$

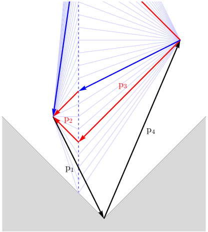
$$T_\Delta(w) = \frac{w^{\Delta-1}}{\Gamma(2\Delta_\phi - \Delta)\Gamma(\Delta)} {}_2F_1(\Delta - 2\Delta_\phi + 1, \Delta; 2\Delta; w)$$

Matching discontinuities in 2d

Kinematics of discontinuity fixed by p_1 and p_4:

$$w \equiv -\frac{p_1^+}{p_4^+} = \frac{p_1^2}{s} = \frac{t}{p_4^2} \quad \in (0,1)$$

$$\bar{w} \equiv -\frac{p_4^-}{p_1^-} = \frac{p_4^2}{s} = \frac{t}{p_1^2} \quad \in (0,1)$$



$$\operatorname*{disc}_{\vec{q}=\vec{q}_*} \int \frac{dq^0}{q^0 - i} \langle\langle \widetilde{\phi}(p_1)\widetilde{\phi}(p-q)\widetilde{\phi}(p+q)\widetilde{\phi}(p_4)\rangle\rangle \propto \sum_{\mathcal{O}} C^2_{\phi\phi\mathcal{O}} S_h(w) S_{\bar{h}}(\bar{w})$$

$$= \operatorname*{disc}_{\vec{q}=\vec{q}_*} \int \frac{dq^0}{q^0 - i} \langle\langle \widetilde{\phi}(p_1)\widetilde{\phi}(p+q)\widetilde{\phi}(p-q)\widetilde{\phi}(p_4)\rangle\rangle \propto \sum_{\mathcal{O}} C^2_{\phi\phi\mathcal{O}} T_h(w) T_{\bar{h}}(\bar{w})$$

$$S_\Delta(w) = \frac{\Gamma(2\Delta)}{\Gamma(\Delta)^2\Gamma(2\Delta_\phi)} w^{2\Delta_\phi - 1} {}_2F_1(1-\Delta, \Delta; 2\Delta_\phi; w)$$

$$T_\Delta(w) = \frac{w^{\Delta-1}}{\Gamma(2\Delta_\phi - \Delta)\Gamma(\Delta)} {}_2F_1(\Delta - 2\Delta_\phi + 1, \Delta; 2\Delta; w)$$

Perspectives in higher d

What is still valid in $d > 2$?

- Causality $\Rightarrow$ analyticity
- Discontinuity of the blocks at double light cone crossing
- Residue is ${}_2F_1$ hypergeometric of ratios of momenta

What changes in $d > 2$?

- Discontinuous point $\rightarrow$ discontinuous locus
- Kinematic variables are different in the s and t channel:

$$\left(\frac{p_1^2}{s}, \frac{p_4^2}{s}, \cos\theta_s\right) \quad \leftrightarrow \quad \left(\frac{t}{p_1^2}, \frac{t}{p_4^2}, \cos\theta_t\right)$$

- Spin of internal operator
(but characterized by integer ℓ for external scalars)

Conformal blocks

More work needed to fit on a slide!

Generalized free field theory

At double-twist dimensions $\Delta = 2\Delta_\phi + \ell + 2n$,

- t-channel blocks are zero
- s-channel blocks are a sum of products of polynomials

But spin complicates the matter...

- Gegenbauer $C_m^{(d-3)/2}(\cos\theta_s)$ of degree $m = 0, \ldots, \ell$
- Jacobi $P_{n+j}\left(1 - 2p_1^2/s\right)$ and $P_{n+\bar{\jmath}}\left(1 - 2p_4^2/s\right)$ with $j, \bar{\jmath} = 0, \ldots, \ell - m$ → possible improvements?

Orthogonality can be used to recover OPE coefficients

Fitzpatrick, Kaplan 1112.4845

Conclusions

Summary

- A new bootstrap equation in 2 and higher d, with conformal blocks known in closed form in any d!
- A nice basis of functionals with zeros at double-twist dimensions
- All results also for distinct external dimensions

Outlook

- Analytical bootstrap in a neighborhood of GFF (e.g. weakly relevant flows, AdS duals, ...)
- Build optimal functionals, numerically and analytically
- Other numerical approaches using asymmetry